

Symmetric Hamiltonian of the Garnier system and its degenerate systems in two variables

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Abstract

We present *symmetric Hamiltonians* for the degenerate Garnier systems in two variables. For these symmetric Hamiltonians, we make the symmetry and holomorphy conditions, and we also make a generalization of these systems involving symmetry and holomorphy conditions inductively. We also show the confluence process among each system by taking the coupling confluence process of the Painlevé systems.

Key Words and Phrases. Bäcklund transformation, Birational transformation, Holomorphy condition, Painlevé equations, Garnier system.

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1 Introduction

In this paper, we consider the Garnier system $G(1,1,1,1)$ and its degenerate systems in two variables, which are completely integrable Hamiltonian systems of the form

$$\begin{aligned} dx &= \frac{\partial H_1}{\partial y} dt + \frac{\partial H_2}{\partial y} ds, & dy &= -\frac{\partial H_1}{\partial x} dt - \frac{\partial H_2}{\partial x} ds, \\ dz &= \frac{\partial H_1}{\partial w} dt + \frac{\partial H_2}{\partial w} ds, & dw &= -\frac{\partial H_1}{\partial z} dt - \frac{\partial H_2}{\partial z} ds. \end{aligned} \quad (1)$$

The Hamiltonians H_1, H_2 are polynomial with respect to x, y, z, w whose coefficients are rational functions of t, s .

As is explained in [2], these systems are obtained as monodromy preserving deformations equations of second-order linear ordinary differential equations with regular or irregular singular points and apparent singular points. Let us assign 1 to a regular singular point and $r+1$ to an irregular singular point of Poincaré rank r .

K. Kimura also showed the confluence process among each system through the confluence process of the second-order linear ordinary differential equations.

In this paper, we present the polynomial Hamiltonian system (1) with *symmetric Hamiltonians*

$$H_1 := H_*(x, y, t; \alpha_0, \dots) + R(x, y, z, w, t, s; \alpha_0, \dots) \in \mathbb{C}(t, s)[x, y, z, w], \quad H_2 = \pi(H_1), \quad (2)$$

where the transformation π is given by

$$\pi : (x, y, z, w, t, s) \rightarrow (z, w, x, y, s, t), \quad (\pi)^2 = 1 \quad (3)$$

with some parameter's change, and the symbol $H_*(x, y, t; \alpha_0, \dots)$ denotes one of the Painlevé Hamiltonians (see [7]). The birational symplectic transformations take known Hamiltonian systems (see [2]) to the symmetric Hamiltonian systems. These symmetric Hamiltonians are new.

For the symmetric Hamiltonian systems, we make the symmetry and holomorphy conditions.

As is well-known, the degeneration from P_{VI} to P_V (see [7]) is given by

$$\begin{aligned} \alpha_0 &= \varepsilon^{-1}, \quad \alpha_1 = A_3, \quad \alpha_3 = A_0 - A_2 - \varepsilon^{-1}, \quad \alpha_4 = A_1 \\ t &= 1 + \varepsilon T, \quad (x-1)(X-1) = 1, \quad (x-1)y + (X-1)Y = -A_2. \end{aligned}$$

Notice that

$$A_0 + A_1 + A_2 + A_3 = \alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1$$

and the change of variables from (x, y) to (X, Y) is symplectic.

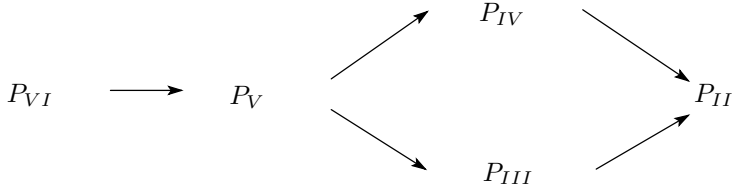
As the fourth-order analogue of the above confluence process, we consider the following coupling confluence process from the Garnier system $G(1,1,1,1,1)$ with symmetric Hamiltonians (see Theorem 2.3). We take the following coupling confluence process $P_{VI} \rightarrow P_V$ for the coordinate system (x, y) and (z, w) of this system. Precisely speaking, for the Garnier system $G(1,1,1,1,1)$, we make the change of parameters and variables

$$\begin{aligned} \alpha_1 &= A_3 + A_5 - 1, & \alpha_2 &= A_1, & \alpha_3 &= 1 - A_5, \\ \alpha_4 &= A_2, & \alpha_5 &= A_4 + A_5 - \frac{1}{\varepsilon}, & \alpha_6 &= 1 - A_3 - A_5 + \frac{1}{\varepsilon}, \end{aligned} \quad (4)$$

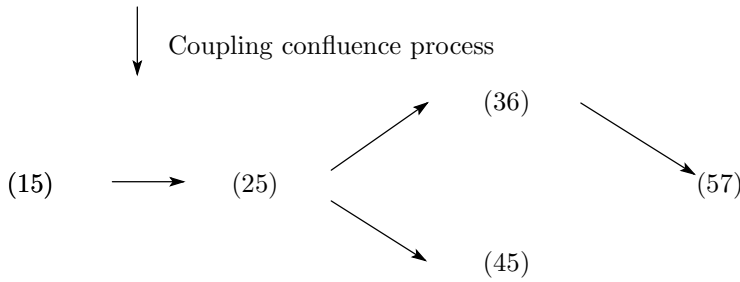
$$\begin{aligned} T &= \frac{t-1}{\varepsilon}, & X &= \frac{x}{x-1}, & Y &= -(x-1)\{(x-1)y + \alpha_2\}, \\ S &= \frac{s-1}{\varepsilon}, & Z &= \frac{z}{z-1}, & W &= -(z-1)\{(z-1)w + \alpha_4\}. \end{aligned} \quad (5)$$

from $\alpha_1, \alpha_2, \dots, \alpha_6, t, s, x, y, z, w$ to $A_1, \dots, A_5, \varepsilon, T, S, X, Y, Z, W$. Then this system can also be written in the new variables T, S, X, Y, Z, W and parameters $A_1, \dots, A_5, \varepsilon$ as a Hamiltonian system. This new system tends to the degenerate Garnier system $G(1,1,1,2)$ with symmetric Hamiltonians (see Theorem 5.1) as $\varepsilon \rightarrow 0$ (see in Section 3).

We also show the confluence process among each system by taking the coupling confluence process of the Painlevé systems (see Figure 1).



Confluence process of the Painlevé equations



Confluence process of the Garnier system in two variables

Figure 1:

In this paper, we do not consider the coupling confluence process $P_{III} \rightarrow P_{II}$. These coupling confluence processes are new.

Moreover, we can make a generalization of these systems involving symmetry and holomorphy conditions without monodromy preserving deformations equations of second-order linear ordinary differential equations. The Hamiltonian system obtained by this process is generally rational functions of t, s, x, y, z, w, q, p , and is not symmetric. However, our Hamiltonians are all polynomials

with respect to x, y, z, w, q, p and symmetric. The system is explicitly given by

$$\begin{aligned} dx &= \frac{\partial H_1}{\partial y} dt + \frac{\partial H_2}{\partial y} ds + \frac{\partial H_3}{\partial y} du, & dy &= -\frac{\partial H_1}{\partial x} dt - \frac{\partial H_2}{\partial x} ds - \frac{\partial H_3}{\partial x} du, \\ dz &= \frac{\partial H_1}{\partial w} dt + \frac{\partial H_2}{\partial w} ds + \frac{\partial H_3}{\partial w} du, & dw &= -\frac{\partial H_1}{\partial z} dt - \frac{\partial H_2}{\partial z} ds - \frac{\partial H_3}{\partial z} du, \\ dq &= \frac{\partial H_1}{\partial p} dt + \frac{\partial H_2}{\partial p} ds + \frac{\partial H_3}{\partial p} du, & dp &= -\frac{\partial H_1}{\partial q} dt - \frac{\partial H_2}{\partial q} ds - \frac{\partial H_3}{\partial q} du \end{aligned} \quad (6)$$

with the symmetric polynomial Hamiltonians $H_i \in \mathbb{C}(t, s, u)[x, y, z, w, q, p]$ ($i = 1, 2, 3$)

$$\begin{aligned} H_1 &:= H_*(x, y, t; \alpha_0, \dots) + R(x, y, z, w, t, s; \alpha_0, \dots) + R(x, y, q, p, t, u; \alpha_0, \dots) \\ H_2 &= \pi(H_1), \quad H_3 = (\pi \circ \pi)(H_1), \end{aligned} \quad (7)$$

where the transformation π is explicitly given by

$$\pi : (x, y, z, w, q, p, t, s, u) \rightarrow (z, w, q, p, x, y, s, u, t), \quad (\pi)^3 = 1 \quad (8)$$

with some parameter's change, and the symbol $H_*(x, y, t; \alpha_0, \dots)$ denotes one of the Painlevé Hamiltonians.

Moreover, we can make the symmetry and holomorphy conditions inductively. These Hamiltonian systems in three variables are new.

2 Garnier system in two variables

The Garnier system in two variables is equivalent to the Hamiltonian system (see [8])

$$\begin{aligned} dx &= \frac{\partial H_1}{\partial y} dt + \frac{\partial H_2}{\partial y} ds, & dy &= -\frac{\partial H_1}{\partial x} dt - \frac{\partial H_2}{\partial x} ds, \\ dz &= \frac{\partial H_1}{\partial w} dt + \frac{\partial H_2}{\partial w} ds, & dw &= -\frac{\partial H_1}{\partial z} dt - \frac{\partial H_2}{\partial z} ds \end{aligned} \quad (9)$$

with the polynomial Hamiltonians

$$\begin{aligned} H_1 &= H_{VI}(x, y, t; 1 - 2\alpha_1 - \alpha_2 - \alpha_3 - \alpha_5, \alpha_2, \alpha_1, \alpha_5, \alpha_3) \\ &\quad - \frac{\alpha_4 s}{t(t-s)} xy - \frac{\alpha_3(s-1)}{(t-1)(t-s)} zw + \frac{2(s-1)xyzw}{(t-1)(t-s)} \\ &\quad - \frac{t(xy - \alpha_3)yz + s(zw - \alpha_4)xw}{t(t-s)} + \frac{\{2(xy + \alpha_1) + zw + \alpha_2\}xzw}{t(t-1)}, \end{aligned} \quad (10)$$

$$H_2 = \pi(H_1), \quad (11)$$

where the transformation π is explicitly given by

$$\pi : (x, y, z, w, t, s; \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) \rightarrow (z, w, x, y, s, t; \alpha_1, \alpha_2, \alpha_4, \alpha_3, \alpha_5, \alpha_6), \quad (12)$$

and the symbol $H_{VI}(x, y, t; \alpha_0, \alpha_1, \dots, \alpha_4)$ denotes the sixth Painlevé Hamiltonian given by

$$\begin{aligned} &H_{VI}(x, y, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) \\ &= \frac{1}{t(t-1)} [y^2(x-t)(x-1)x - \{(\alpha_0-1)(x-1)x + \alpha_3(x-t)x \\ &\quad + \alpha_4(x-t)(x-1)\}y + \alpha_2(\alpha_1 + \alpha_2)(x-t)] \quad (\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1). \end{aligned} \quad (13)$$

Theorem 2.1. *Let us consider a polynomial Hamiltonian system with Hamiltonian $H_i \in \mathbb{C}(t, s)[x, y, z, w]$ ($i = 1, 2$). We assume that*

(A1) *$\deg(H_i) = 5$ with respect to x, y, z, w .*

(A2) *This system becomes again a polynomial Hamiltonian system in each coordinate system $\{U_j, (x_j, y_j, z_j, w_j)\}$ ($j = 1, 2, \dots, 6$):*

$$U_j = \mathbb{C}^4 \ni (x_j, y_j, z_j, w_j) \quad (j = 1, 2, \dots, 6),$$

via the following birational and symplectic transformations

- 1) $x_1 = \frac{1}{x}$, $y_1 = -x(xy + zw + \alpha_1)$, $z_1 = \frac{z}{x}$, $w_1 = xw$,
- 2) $x_2 = \frac{1}{x}$, $y_2 = -x(xy + zw + \alpha_1 + \alpha_2)$, $z_2 = \frac{z}{x}$, $w_2 = xw$,
- 3) $x_3 = -y(xy - \alpha_3)$, $y_3 = \frac{1}{y}$, $z_3 = z$, $w_3 = w$,
- 4) $x_4 = x$, $y_4 = y$, $z_4 = -w(zw - \alpha_4)$, $w_4 = \frac{1}{w}$,
- 5) $x_5 = -((x + z - 1)y - \alpha_5)y$, $y_5 = \frac{1}{y}$, $z_5 = z$, $w_5 = w - y$,
- 6) $x_6 = -((x + tz/s - t)y - \alpha_6)y$, $y_6 = \frac{1}{y}$, $z_6 = z$, $w_6 = w - ty/s$.

Then such a system coincides with the system (1).

Theorem 2.2. On each affine open set $(x_j, y_j, z_j, w_j) \in U_j \times B$ in Theorem 1.1, the Hamiltonians H_{j1} and H_{j2} on $U_j \times B$ are expressed as a polynomial in x_j, y_j, z_j, w_j and a rational function t and s , and satisfies the following conditions:

$$\begin{aligned}
& dx \wedge dy + dz \wedge dw - dH_1 \wedge dt - dH_2 \wedge ds \\
& = dx_j \wedge dy_j + dz_j \wedge dw_j - dH_{j1} \wedge dt - dH_{j2} \wedge ds \quad (j = 1, 2, \dots, 5), \\
& dx \wedge dy + dz \wedge dw - d(H_1 - (1 - z/s)y) \wedge dt - d(H_2 - (1 - x/t)w) \wedge ds \\
& = dx_6 \wedge dy_6 + dz_6 \wedge dw_6 - dH_{61} \wedge dt - dH_{62} \wedge ds.
\end{aligned}$$

Theorem 2.3. The birational and symplectic transformation

$$S : (x, y, z, w; t, s) \rightarrow (x + (zw + \alpha_1)/y, y, w/y, -zy, t, t/s), \quad (14)$$

takes the system (1) to the Hamiltonian system

$$\begin{aligned}
dx &= \frac{\partial H_1}{\partial y} dt + \frac{\partial H_2}{\partial y} ds, & dy &= -\frac{\partial H_1}{\partial x} dt - \frac{\partial H_2}{\partial x} ds, \\
dz &= \frac{\partial H_1}{\partial w} dt + \frac{\partial H_2}{\partial w} ds, & dw &= -\frac{\partial H_1}{\partial z} dt - \frac{\partial H_2}{\partial z} ds
\end{aligned} \quad (15)$$

with the polynomial Hamiltonians

$$\begin{aligned}
H_1 &= H_{VI}(x, y, t; 1 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_5, -\alpha_1 - \alpha_2, \alpha_2, \alpha_1 + \alpha_5, \alpha_1 + \alpha_3) \\
&+ \frac{\alpha_4 xy}{(t - s)} + \frac{\alpha_2(s - 1)zw + 2(s - 1)xyzw}{(t - 1)(t - s)} + \frac{\{ty + (xy + \alpha_2)x\}zw}{t(t - 1)} \\
&- \frac{t(zw + \alpha_4)yz + s(xy + \alpha_2)xw}{t(t - s)},
\end{aligned} \quad (16)$$

$$H_2 = \pi(H_1), \quad (17)$$

where the transformation π is explicitly given by

$$\pi : (x, y, z, w; t, s, \alpha_1, \alpha_2, \dots, \alpha_6) \rightarrow (z, w, x, y; s, t, \alpha_1, \alpha_4, \alpha_3, \alpha_2, \alpha_5, \alpha_6). \quad (18)$$

This transformation can be regarded as a generalization of Okamoto-transformation of the sixth Painlevé system.

Remark 2.1. The system (1) is not invariant under S .

Theorem 2.4. Let us consider a polynomial Hamiltonian system with Hamiltonian $H_i \in \mathbb{C}(t, s)[x, y, z, w]$ ($i = 1, 2$). We assume that
(A1) $\deg(H_i) = 5$ with respect to x, y, z, w .

(A2) This system becomes again a polynomial Hamiltonian system in each coordinate system $\{U_j, (x_j, y_j, z_j, w_j)\}$ ($j = 1, 2, \dots, 6$):

$$U_j = \mathbb{C}^4 \ni (x_j, y_j, z_j, w_j) \quad (j = 1, 2, \dots, 6),$$

via the following birational and symplectic transformations

$$\begin{aligned} 1) x_1 &= \frac{1}{x}, \quad y_1 = -x(xy + \alpha_2), \quad z_1 = z, \quad w_1 = w, \\ 2) x_2 &= 1/x, \quad y_2 = -x(xy + zw - \alpha_1), \quad z_2 = \frac{z}{x}, \quad w_2 = xw, \\ 3) x_3 &= x, \quad y_3 = y, \quad z_3 = 1/z, \quad w_3 = -(zw + \alpha_4)z, \\ 4) x_4 &= -(xy + zw - (\alpha_1 + \alpha_3))y, \quad y_4 = 1/y, \quad z_4 = zy, \quad w_4 = w/y, \\ 5) x_5 &= -((x-1)y + (z-1)w - (\alpha_1 + \alpha_5))y, \quad y_5 = 1/y, \quad z_5 = (z-1)y, \quad w_5 = w/y, \\ 6) x_6 &= -((x-t)y + (z-s)w - (\alpha_1 + \alpha_6))y, \quad y_6 = 1/y, \quad z_6 = (z-s)y, \quad w_6 = w/y. \end{aligned}$$

Then such a system coincides with the system (??).

Theorem 2.5. On each affine open set $(x_j, y_j, z_j, w_j) \in U_j \times B$ in Theorem 2.3, the Hamiltonians H_{j1} and H_{j2} on $U_j \times B$ are expressed as a polynomial in x_j, y_j, z_j, w_j and a rational function in t and s , and satisfies the following conditions:

$$\begin{aligned} dx \wedge dy + dz \wedge dw - dH_1 \wedge dt - dH_2 \wedge ds \\ = dx_j \wedge dy_j + dz_j \wedge dw_j - dH_{j1} \wedge dt - dH_{j2} \wedge ds \quad (j = 1, 2, \dots, 5), \\ dx \wedge dy + dz \wedge dw - d(H_1 - y) \wedge dt - d(H_2 - w) \wedge ds \\ = dx_6 \wedge dy_6 + dz_6 \wedge dw_6 - dH_{61} \wedge dt - dH_{62} \wedge ds. \end{aligned}$$

Theorem 2.6. The system (15) is invariant under the following transformations: with the notation $(*) = (x, y, z, w, t, s; \alpha_1, \alpha_2, \dots, \alpha_6)$,

$$\begin{aligned} u_1 : (*) &\rightarrow (x + \alpha_2/y, y, z, w, t, s; \alpha_1 + \alpha_1, -\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6), \\ u_2 : (*) &\rightarrow (x, y, z + \alpha_4/w, w, t, s; \alpha_1 + \alpha_4, \alpha_2, \alpha_3, -\alpha_4, \alpha_5, \alpha_6), \\ u_3 : (*) &\rightarrow \left(\frac{x(xy + zw - \alpha_1)}{(xy + zw - \alpha_1 - \alpha_3)}, \frac{y(xy + zw - \alpha_1 - \alpha_3)}{(xy + zw - \alpha_1)}, \frac{z(xy + zw - \alpha_1)}{(xy + zw - \alpha_1 - \alpha_3)}, \right. \\ &\quad \left. \frac{w(xy + zw - \alpha_1 - \alpha_3)}{(xy + zw - \alpha_1)}, t, s; \alpha_1 + \alpha_3, \alpha_2, -\alpha_3, \alpha_4, \alpha_5, \alpha_6 \right), \\ \varphi_1 : (*) &\rightarrow (1/x, -(xy + \alpha_2)x, 1/z, -(zw + \alpha_4)z, 1/t, 1/s; \\ &\quad -\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4, \alpha_2, \alpha_3, \alpha_4, 1 - \alpha_6, 1 - \alpha_5), \\ \varphi_2 : (*) &\rightarrow (1 - x, -y, 1 - z, -w, 1 - t, 1 - s; \alpha_1, \alpha_2, \alpha_5, \alpha_4, \alpha_3, \alpha_6), \\ \varphi_3 : (*) &\rightarrow \left(\frac{t-x}{t-1}, -(t-1)y, \frac{s-z}{s-1}, -(s-1)w, \frac{t}{t-1}, \frac{s}{s-1}; \alpha_1, \alpha_2, \alpha_6, \alpha_4, \alpha_5, \alpha_3 \right). \end{aligned}$$

3 A generalization of the system (15) to three variables

In this section, we present a generalization of the system (15) to three variables t, s and u , which is equivalent to the polynomial Hamiltonian system

$$\begin{aligned} dx &= \frac{\partial H_1}{\partial y} dt + \frac{\partial H_2}{\partial y} ds + \frac{\partial H_3}{\partial y} du, & dy &= -\frac{\partial H_1}{\partial x} dt - \frac{\partial H_2}{\partial x} ds - \frac{\partial H_3}{\partial x} du, \\ dz &= \frac{\partial H_1}{\partial w} dt + \frac{\partial H_2}{\partial w} ds + \frac{\partial H_3}{\partial w} du, & dw &= -\frac{\partial H_1}{\partial z} dt - \frac{\partial H_2}{\partial z} ds - \frac{\partial H_3}{\partial z} du, \\ dq &= \frac{\partial H_1}{\partial p} dt + \frac{\partial H_2}{\partial p} ds + \frac{\partial H_3}{\partial p} du, & dp &= -\frac{\partial H_1}{\partial q} dt - \frac{\partial H_2}{\partial q} ds - \frac{\partial H_3}{\partial q} du \end{aligned} \tag{19}$$

with the symmetric Hamiltonians $H_i \in \mathbb{C}(t, s, u)[x, y, z, w, q, p]$ ($i = 1, 2, 3$)

$$\begin{aligned} H_1 &= H_{VI}(x, y, t; 1 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_5, -\alpha_1 - \alpha_2, \alpha_2, \alpha_1 + \alpha_5, \alpha_1 + \alpha_3) \\ &\quad + R(x, y, z, w, t, s; \alpha_2, \alpha_4) + R(x, y, q, p, t, u; \alpha_2, \alpha_7), \\ H_2 &= \pi(H_1), \quad H_3 = (\pi \circ \pi)(H_1) \quad (2\alpha_1 + \alpha_2 + \dots + \alpha_7 = 1), \end{aligned} \quad (20)$$

where the transformation π is explicitly given by

$$\pi : (*) \rightarrow (z, w, q, p, x, y, s, u, t; \alpha_1, \alpha_4, \alpha_3, \alpha_7, \alpha_5, \alpha_6, \alpha_2). \quad (21)$$

Here the symbol $(*)$ denotes $(*) := (x, y, z, w, q, p, t, s, u; \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7)$, and the symbol $R(q_l, p_l, q_m, p_m, t_l, t_m; \alpha, \beta)$ is explicitly given by

$$\begin{aligned} &R(q_l, p_l, q_m, p_m, t_l, t_m; \alpha, \beta) \\ &= \frac{\beta q_l p_l}{t_l - t_m} + \frac{\alpha(t_m - 1)q_m p_m}{(t_l - 1)(t_l - t_m)} + \frac{\{t_l p_l + (q_l p_l + \alpha)q_l\}q_m p_m}{t_l(t_l - 1)} \\ &\quad - \frac{t_l(q_m p_m + \beta)p_l q_m + t_m(q_l p_l + \alpha)q_l p_m}{t_l(t_l - t_m)}. \end{aligned}$$

For the Hamiltonian system (9), it is difficult to make a generalization to three variables by a similar way.

Theorem 3.1. *Let us consider a polynomial Hamiltonian system with Hamiltonian $H_i \in \mathbb{C}(t, s, u)[x, y, z, w, q, p]$ ($i = 1, 2, 3$). We assume that*

(A1) *$\deg(H_i) = 5$ with respect to x, y, z, w, q, p .*

(A2) *This system becomes again a polynomial Hamiltonian system in each coordinate system $\{U_j, (x_j, y_j, z_j, w_j, q_j, p_j)\}$ ($j = 1, 2, \dots, 7$):*

$$U_j = \mathbb{C}^6 \ni (x_j, y_j, z_j, w_j, q_j, p_j) \quad (j = 1, 2, \dots, 7),$$

via the following birational and symplectic transformations

$$\begin{aligned} 1) &x_1 = 1/x, \quad y_1 = -x(xy + zw + qp - \alpha_1), \quad z_1 = \frac{z}{x}, \quad w_1 = xw, \quad q_1 = \frac{q}{x}, \quad p_1 = xp, \\ 2) &x_2 = \frac{1}{x}, \quad y_2 = -x(xy + \alpha_2), \quad z_2 = z, \quad w_2 = w, \quad q_2 = q, \quad p_2 = p, \\ 3) &x_3 = -(xy + zw + qp - (\alpha_1 + \alpha_3))y, \quad y_3 = 1/y, \quad z_3 = zy, \quad w_3 = w/y, \\ &q_3 = qy, \quad w_3 = p/y, \\ 4) &x_4 = x, \quad y_4 = y, \quad z_4 = 1/z, \quad w_4 = -(zw + \alpha_4)z, \quad q_4 = q, \quad p_4 = p, \\ 5) &x_5 = -((x - 1)y + (z - 1)w + (q - 1)p - (\alpha_1 + \alpha_5))y, \quad y_5 = 1/y, \\ &z_5 = (z - 1)y, \quad w_5 = w/y, \quad q_5 = (q - 1)y, \quad p_5 = p/y, \\ 6) &x_6 = -((x - t)y + (z - s)w + (q - u)p - (\alpha_1 + \alpha_6))y, \quad y_6 = 1/y, \\ &z_6 = (z - s)y, \quad w_6 = w/y, \quad q_6 = (q - u)y, \quad p_6 = p/y, \\ 7) &x_7 = x, \quad y_7 = y, \quad z_7 = z, \quad w_7 = w, \quad q_7 = 1/q, \quad p_7 = -(qp + \alpha_7)q. \end{aligned}$$

Then such a system coincides with the system (19).

Theorem 3.2. *The system (19) is invariant under the following transformations: with the nota-*

tion $(*) = (x, y, z, w, q, p, t, s, u; \alpha_1, \alpha_2, \dots, \alpha_7)$,

$$\begin{aligned}
u_2 : (*) &\rightarrow (x + \alpha_2/y, y, z, w, q, p, t, s, u; \alpha_1 + \alpha_1, -\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7), \\
u_4 : (*) &\rightarrow (x, y, z + \alpha_4/w, w, q, p, t, s, u; \alpha_1 + \alpha_4, \alpha_2, \alpha_3, -\alpha_4, \alpha_5, \alpha_6, \alpha_7), \\
u_7 : (*) &\rightarrow (x, y, z, w, q, p + \alpha_7/p, t, s, u; \alpha_1 + \alpha_7, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, -\alpha_7), \\
u_3 : (*) &\rightarrow \left(\frac{x(xy + zw + qp - \alpha_1)}{(xy + zw + qp - \alpha_1 - \alpha_3)}, \frac{y(xy + zw + qp - \alpha_1 - \alpha_3)}{(xy + zw + qp - \alpha_1)}, \right. \\
&\quad \frac{z(xy + zw + qp - \alpha_1)}{(xy + zw + qp - \alpha_1 - \alpha_3)}, \frac{w(xy + zw + qp - \alpha_1 - \alpha_3)}{(xy + zw + qp - \alpha_1)}, \\
&\quad \frac{q(xy + zw + qp - \alpha_1)}{(xy + zw + qp - \alpha_1 - \alpha_3)}, \frac{p(xy + zw + qp - \alpha_1 - \alpha_3)}{(xy + zw + qp - \alpha_1)}, \\
&\quad \left. t, s, u; \alpha_1 + \alpha_3, \alpha_2, -\alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7 \right), \\
\varphi_1 : (*) &\rightarrow (1/x, -(xy + \alpha_2)x, 1/z, -(zw + \alpha_4)z, 1/q, -(qp + \alpha_7)q, 1/t, 1/s, 1/u; \\
&\quad -\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_7, \alpha_2, \alpha_3, \alpha_4, 1 - \alpha_6, 1 - \alpha_5, \alpha_7), \\
\varphi_2 : (*) &\rightarrow (1 - x, -y, 1 - z, -w, 1 - q, -p, 1 - t, 1 - s, 1 - u; \\
&\quad \alpha_1, \alpha_2, \alpha_5, \alpha_4, \alpha_3, \alpha_6, \alpha_7), \\
\varphi_3 : (*) &\rightarrow \left(\frac{t - x}{t - 1}, -(t - 1)y, \frac{s - z}{s - 1}, -(s - 1)w, \frac{u - q}{u - 1}, -(u - 1)p, \right. \\
&\quad \left. \frac{t}{t - 1}, \frac{s}{s - 1}, \frac{u}{u - 1}; \alpha_1, \alpha_2, \alpha_6, \alpha_4, \alpha_5, \alpha_3, \alpha_7 \right).
\end{aligned}$$

4 Degenerate Garnier system $G(1, 1, 1, 2)$ in two variables

The degenerate Garnier system $G(1, 1, 1, 2)$ in two variables t, s is equivalent to the Hamiltonian system

$$\begin{aligned}
dx &= \frac{\partial H_1}{\partial y} dt + \frac{\partial H_2}{\partial y} ds, & dy &= -\frac{\partial H_1}{\partial x} dt - \frac{\partial H_2}{\partial x} ds, \\
dz &= \frac{\partial H_1}{\partial w} dt + \frac{\partial H_2}{\partial w} ds, & dw &= -\frac{\partial H_1}{\partial z} dt - \frac{\partial H_2}{\partial z} ds
\end{aligned} \tag{22}$$

with the polynomial Hamiltonians $K_i \in \mathbb{C}(t, s)[x, y, z, w]$ ($i = 1, 2$) (see [1])

$$\begin{aligned}
t^2 K_1 &= x^2(x - t)y^2 + 2x^2zyw + xz(z - s)w^2 \\
&\quad - \{(\alpha_0 + \alpha_2 - 1)x^2 + \alpha_1x(x - t) + \eta(x - t) + \eta tz\}y \\
&\quad - \{(\alpha_0 + \alpha_1 - 1)xz + \alpha_2x(z - s) + \eta(s - 1)z\}w + \nu(\nu + \alpha_3)x, \\
s(s - 1)K_2 &= x^2zy^2 + 2xz(z - s)yw \\
&\quad + \{z(z - 1)(z - s) + \frac{s(s - 1)}{t}xz\}w^2 \\
&\quad - \{(\alpha_0 + \alpha_1 - 1)xz + \alpha_2x(z - s) - \eta(s - 1)z\}y \\
&\quad - \{(\alpha_0 - 1)z(z - 1) + \alpha_1z(z - s) + \alpha_2(z - 1)(z - s) \\
&\quad + \frac{s(s - 1)}{t}(\alpha_2x + \eta z)\}w + \nu(\nu + \alpha_3)z \left(\nu = -\frac{1}{2}(\alpha_0 + \alpha_1 + \alpha_2 - 1 + \alpha_3) \right).
\end{aligned} \tag{23}$$

Theorem 4.1. *The system (22) is invariant under the following transformations: with the notation $(*) = (x, y, z, w, \eta, t, s; \alpha_0, \alpha_1, \dots, \alpha_3, \nu)$,*

$$\begin{aligned}
s_0 : (*) &\rightarrow \left(x, y - \frac{s\alpha_0}{sx + tz - ts}, z, w - \frac{t\alpha_0}{sx + tz - ts}, \eta, t, s; -\alpha_0, \alpha_1, \alpha_2, -\alpha_3, \nu \right), \\
s_1 : (*) &\rightarrow \left(x, y - \frac{\alpha_1}{x} + \frac{\eta(z - 1)}{x^2}, z, w - \frac{\eta}{x}, -\eta, t, s; \alpha_0, -\alpha_1, \alpha_2, -\alpha_3, \nu \right), \\
s_2 : (*) &\rightarrow \left(x, y, z, w - \frac{\alpha_2}{z}, \eta, t, s; \alpha_0, \alpha_1, -\alpha_2, \alpha_3, \nu + \alpha_2 \right), \\
s_3 : (*) &\rightarrow (x, y, z, w, \eta, t, s; \alpha_0, \alpha_1, \alpha_2, -\alpha_3, \nu + \alpha_3).
\end{aligned}$$

Theorem 4.2. *Let us consider a polynomial Hamiltonian system with Hamiltonian $H_i \in \mathbb{C}(t, s)[x, y, z, w]$ ($i = 1, 2$). We assume that*

(A1) *$\deg(H_i) = 5$ with respect to x, y, z, w .*

(A2) *This system becomes again a polynomial Hamiltonian system in each coordinate system $\{U_j, (x_j, y_j, z_j, w_j)\}$ ($j = 1, 2, \dots, 5$):*

$$U_j = \mathbb{C}^4 \ni (x_j, y_j, z_j, w_j) \quad (j = 1, 2, \dots, 5),$$

via the following birational and symplectic transformations

$$\begin{aligned} 1) x_1 &= \frac{1}{x}, & y_1 &= -(xy + zw + \nu + \alpha_3)x, & z_1 &= \frac{z}{x}, & w_1 &= xw, \\ 2) x_2 &= \frac{1}{x}, & y_2 &= -(xy + zw + \nu)x, & z_2 &= \frac{z}{x}, & w_2 &= xw, \\ 3) x_3 &= x, & y_3 &= y - \frac{\alpha_1}{x} + \frac{\eta(z-1)}{x^2}, & z_3 &= z, & w_3 &= w - \frac{\eta}{x}, \\ 4) x_4 &= x, & y_4 &= y, & z_4 &= -(zw - \alpha_2)w, & w_4 &= \frac{1}{w}, \\ 5) x_5 &= -\left(\left(x + \frac{t}{s}z - t\right)y - \alpha_0\right)y, & y_5 &= \frac{1}{y}, & z_5 &= z, & w_5 &= w - \frac{t}{s}y. \end{aligned}$$

Then such a system coincides with the system (22).

5 Symmetric Hamiltonian of the system (22)

In this section, we make *symmetric Hamiltonian* for the degenerate Garnier system $G(1, 1, 1, 2)$ by taking suitable birational and symplectic transformations. By making this symmetric Hamiltonian, we can make a generalization of this system involving symmetry and holomorphy in the next section. We also show the confluence process from the system (15) to this system by taking the coupling confluence process $P_{VI} \rightarrow P_V$ for each coordinate system (x, y) and (z, w) of the system (15), respectively.

Theorem 5.1. *The birational and symplectic transformations with parameter's change:*

$$\begin{aligned} X &:= x(xy + zw + \nu), & Y &:= \frac{1}{x}, & Z &:= xw, & W &:= -\frac{z}{x}, & T &:= -\frac{1}{t}, & S &:= -\frac{s}{t}, \\ \eta &= 1, & (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \nu) &\rightarrow (\alpha_5, \alpha_4 - \alpha_3, \alpha_2, \alpha_1, \alpha_3) \end{aligned} \quad (24)$$

takes the system (22) to the polynomial Hamiltonian system

$$\begin{aligned} dx &= \frac{\partial H_1}{\partial y} dt + \frac{\partial H_2}{\partial y} ds, & dy &= -\frac{\partial H_1}{\partial x} dt - \frac{\partial H_2}{\partial x} ds, \\ dz &= \frac{\partial H_1}{\partial w} dt + \frac{\partial H_2}{\partial w} ds, & dw &= -\frac{\partial H_1}{\partial z} dt - \frac{\partial H_2}{\partial z} ds \end{aligned} \quad (25)$$

with the symmetric Hamiltonians $H_i \in \mathbb{C}(t, s)[x, y, z, w]$ ($i = 1, 2$)

$$\begin{aligned} H_1 &= H_V(x, y, t; \alpha_3, \alpha_1, \alpha_4) - \frac{\alpha_2 sxy}{t(s-t)} - \frac{\alpha_1 zw}{s-t} \\ &\quad + \frac{tx^2yw + tyzw + \alpha_1 txw + syz^2w - syzw + \alpha_2 syz - 2txyzw}{t(s-t)}, \\ H_2 &= \pi(H_1) \quad (\alpha_1 + \alpha_2 + \dots + \alpha_5 = 1), \end{aligned} \quad (26)$$

Here, for notational convenience, we have renamed X, Y, Z, W, T, S to x, y, z, w, t, s (which are not the same as the previous x, y, z, w, t, s). The transformation π is explicitly given by

$$\pi : (x, y, z, w, t, s; \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \rightarrow (z, w, x, y, s, t; \alpha_2, \alpha_1, \alpha_3, \alpha_4, \alpha_5). \quad (27)$$

We note that the Hamiltonian H_1 involves the Painlevé V Hamiltonian H_V given by

$$H_V(x, y, t; \alpha_1, \alpha_2, \alpha_3) = \frac{x(x-1)y(y+t) + \alpha_2 tx - \alpha_3 xy - \alpha_1 y(x-1)}{t}. \quad (28)$$

Theorem 5.2. *Let us consider a polynomial Hamiltonian system with Hamiltonian $H_i \in \mathbb{C}(t, s)[x, y, z, w]$ ($i = 1, 2$). We assume that*

(A1) *$\deg(H_i) = 5$ with respect to x, y, z, w .*

(A2) *This system becomes again a polynomial Hamiltonian system in each coordinate system $\{U_j, (x_j, y_j, z_j, w_j)\}$ ($j = 1, 2, \dots, 5$):*

$$U_j = \mathbb{C}^4 \ni (x_j, y_j, z_j, w_j) \quad (j = 1, 2, \dots, 5),$$

via the following birational and symplectic transformations

$$\begin{aligned} 1) x_1 &= \frac{1}{x}, & y_1 &= -(yx + \alpha_1)x, & z_1 &= z, & w_1 &= w, \\ 2) x_2 &= x, & y_2 &= y, & z_2 &= \frac{1}{z}, & w_2 &= -(zw + \alpha_2)z, \\ 3) x_3 &= -(yx + wz - \alpha_3)y, & y_3 &= \frac{1}{y}, & z_3 &= zy, & w_3 &= \frac{w}{y}, \\ 4) x_4 &= -((x-1)y + (z-1)w - \alpha_4)y, & y_4 &= \frac{1}{y}, & z_4 &= (z-1)y, & w_4 &= \frac{w}{y}, \\ 5) x_5 &= \frac{1}{x}, & y_5 &= -((y + tw/s + t)x + \alpha_5)x, & z_5 &= z - tx/s, & w_5 &= w. \end{aligned}$$

Then such a system coincides with the system (25).

Theorem 5.3. *The system (25) is invariant under the following transformations: with the notation $(*) = (x, y, z, w, t, s; \alpha_1, \alpha_2, \dots, \alpha_5)$,*

$$\begin{aligned} s_1 : (*) &\rightarrow \left(x + \frac{\alpha_1}{y}, y, z, w, t, s; -\alpha_1, \alpha_2, \alpha_3 + \alpha_1, \alpha_4 + \alpha_1, \alpha_5 \right), \\ s_2 : (*) &\rightarrow \left(x, y, z + \frac{\alpha_2}{w}, w, t, s; \alpha_1, -\alpha_2, \alpha_3 + \alpha_2, \alpha_4 + \alpha_2, \alpha_5 \right), \\ s_5 : (*) &\rightarrow \left(x + \frac{\alpha_5}{y + tw/s + t}, y, z + \frac{t\alpha_5}{s(y + tw/s + t)}, w, t, s; \alpha_1, \alpha_2, \alpha_3 + \alpha_5, \alpha_4 + \alpha_5, -\alpha_5 \right), \\ \pi_1 : (*) &\rightarrow (z, w, x, y, s, t; \alpha_2, \alpha_1, \alpha_3, \alpha_4, \alpha_5), \\ \pi_2 : (*) &\rightarrow (1 - x, -y, 1 - z, -w, -t, -s; \alpha_1, \alpha_2, \alpha_4, \alpha_3, \alpha_5), \\ \pi_3 : (*) &\rightarrow \left(-\frac{xy + zw - \alpha_3}{tx}, \frac{tx(xy + \alpha_1)}{xy + zw - \alpha_3}, -\frac{xy + zw - \alpha_3}{sz}, \frac{sz(zw + \alpha_2)}{xy + zw - \alpha_3}, -t, -s; \right. \\ &\quad \left. \alpha_1, \alpha_2, \alpha_4 + \alpha_5 - 1, \alpha_3 + \alpha_5, 1 - \alpha_5 \right), \\ \pi_4 : (*) &\rightarrow \left(-\frac{y(x-1) + w(z-1) - \alpha_4}{t(x-1)}, \frac{t(x-1)((x-1)y + \alpha_1)}{y(x-1) + w(z-1) - \alpha_4}, \right. \\ &\quad \left. -\frac{y(x-1) + w(z-1) - \alpha_4}{s(z-1)}, \frac{s(z-1)((z-1)w + \alpha_2)}{y(x-1) + w(z-1) - \alpha_4}, t, s; \right. \\ &\quad \left. \alpha_1, \alpha_2, \alpha_3 + \alpha_5 - 1, \alpha_4 + \alpha_5, 1 - \alpha_5 \right). \end{aligned}$$

Proposition 5.1. *The transformation π_3 can be obtained by composing the following transformations:*

Step 1: *We transform the system (25) by the following birational and symplectic transformation*

$$g_1 : (x, y, z, w) \rightarrow (x, y + (zw - \alpha_3)/x, z/x, xw).$$

Step 2: *We then transform the system obtained by Step 1 by the following birational and symplectic transformation*

$$g_2 : (x, y, z, w) \rightarrow (x - (zw - \alpha_1 - \alpha_3)/y, y, z/y, wy).$$

Step 3: *We then transform the system obtained by Step 2 by the following birational and symplectic transformation*

$$g_3 : (*) \rightarrow (y, -x, 1/z, -(zw + \alpha_2)z).$$

Step 4: *We then transform the system obtained by Step 3 by the following birational and symplectic transformation*

$$g_4 : (*) \rightarrow (-x/t, -ty, -z/s, -sw, -t, -s).$$

As is well-known, the degeneration from P_{VI} to P_V (see [7]) is given by

$$\begin{aligned}\alpha_0 &= \varepsilon^{-1}, \quad \alpha_1 = A_3, \quad \alpha_3 = A_0 - A_2 - \varepsilon^{-1}, \quad \alpha_4 = A_1 \\ t &= 1 + \varepsilon T, \quad (x-1)(X-1) = 1, \quad (x-1)y + (X-1)Y = -A_2.\end{aligned}$$

Notice that

$$A_0 + A_1 + A_2 + A_3 = \alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1$$

and the change of variables from (x, y) to (X, Y) is symplectic.

As the fourth-order analogue of the above confluence process, we consider the following coupling confluence process from the system (15). We take the following coupling confluence process $P_{VI} \rightarrow P_V$ for each coordinate system (x, y) and (z, w) of the system (15).

Theorem 5.4. *For the system (15), we make the change of parameters and variables*

$$\begin{aligned}\alpha_1 &= A_3 + A_5 - 1, \quad \alpha_2 = A_1, \quad \alpha_3 = 1 - A_5, \\ \alpha_4 &= A_2, \quad \alpha_5 = A_4 + A_5 - \frac{1}{\varepsilon}, \quad \alpha_6 = 1 - A_3 - A_5 + \frac{1}{\varepsilon},\end{aligned}\tag{29}$$

$$\begin{aligned}T &= \frac{t-1}{\varepsilon}, \quad S = \frac{s-1}{\varepsilon}, \quad X = \frac{x}{x-1}, \quad Z = \frac{z}{z-1}, \\ Y &= -(x-1)\{(x-1)y + \alpha_2\}, \quad W = -(z-1)\{(z-1)w + \alpha_4\}\end{aligned}\tag{30}$$

from $\alpha_1, \alpha_2, \dots, \alpha_6, t, s, x, y, z, w$ to $A_1, \dots, A_5, \varepsilon, T, S, X, Y, Z, W$. Then this system can also be written in the new variables T, S, X, Y, Z, W and parameters $A_1, \dots, A_5, \varepsilon$ as a Hamiltonian system. This new system tends to the system (25) as $\varepsilon \rightarrow 0$.

6 A generalization of the system (25) to three variables

In this section, we present a generalization of the system (25) to three variables t, s and u , which is equivalent to the polynomial Hamiltonian system

$$\begin{aligned}dx &= \frac{\partial H_1}{\partial y}dt + \frac{\partial H_2}{\partial y}ds + \frac{\partial H_3}{\partial y}du, & dy &= -\frac{\partial H_1}{\partial x}dt - \frac{\partial H_2}{\partial x}ds - \frac{\partial H_3}{\partial x}du, \\ dz &= \frac{\partial H_1}{\partial w}dt + \frac{\partial H_2}{\partial w}ds + \frac{\partial H_3}{\partial w}du, & dw &= -\frac{\partial H_1}{\partial z}dt - \frac{\partial H_2}{\partial z}ds - \frac{\partial H_3}{\partial z}du, \\ dq &= \frac{\partial H_1}{\partial p}dt + \frac{\partial H_2}{\partial p}ds + \frac{\partial H_3}{\partial p}du, & dp &= -\frac{\partial H_1}{\partial q}dt - \frac{\partial H_2}{\partial q}ds - \frac{\partial H_3}{\partial q}du\end{aligned}\tag{31}$$

with the symmetric Hamiltonians $H_i \in \mathbb{C}(t, s, u)[x, y, z, w, q, p]$ ($i = 1, 2, 3$)

$$\begin{aligned}H_1 &= H_V(x, y, t; \alpha_4, \alpha_1, \alpha_5) \\ &\quad + R(x, y, z, w, t, s; \alpha_1, \alpha_2) + R(x, y, q, p, t, u; \alpha_1, \alpha_3), \\ H_2 &= \pi(H_1), \quad H_3 = (\pi \circ \pi)(H_1) \quad (\alpha_1 + \alpha_2 + \dots + \alpha_6 = 1),\end{aligned}\tag{32}$$

where the transformation π is explicitly given by

$$\pi : (*) \rightarrow (z, w, q, p, x, y, s, u, t; \alpha_2, \alpha_3, \alpha_1, \alpha_4, \alpha_5, \alpha_6).\tag{33}$$

Here the symbol $(*)$ denotes $(*) := (x, y, z, w, q, p, t, s, u; \alpha_1, \dots, \alpha_6)$, and the symbol $R(q_l, p_l, q_m, p_m, t_l, t_m; \alpha, \beta)$ is explicitly given by

$$\begin{aligned}&R(q_l, p_l, q_m, p_m, t_l, t_m; \alpha, \beta) \\ &= \frac{\beta t_m q_l p_l}{t_l(t_l - t_m)} + \frac{\alpha q_m p_m}{t_l - t_m} \\ &\quad - \frac{t_l q_l^2 p_l p_m + t_l p_l q_m p_m + \alpha t_l q_l p_m + t_m p_l q_m^2 p_m - t_m p_l q_m p_m + \beta t_m p_l q_m - 2 t_l q_l p_l q_m p_m}{t_l(t_l - t_m)}.\end{aligned}$$

Theorem 6.1. *Let us consider a polynomial Hamiltonian system with Hamiltonian $H_i \in \mathbb{C}(t, s, u)[x, y, z, w, q, p]$ ($i = 1, 2, 3$). We assume that*

(A1) *$\deg(H_i) = 5$ with respect to x, y, z, w, q, p .*

(A2) *This system becomes again a polynomial Hamiltonian system in each coordinate system $\{U_j, (x_j, y_j, z_j, w_j, q_j, p_j)\}$ ($j = 1, 2, \dots, 6$):*

$$U_j = \mathbb{C}^6 \ni (x_j, y_j, z_j, w_j, q_j, p_j) \quad (j = 1, 2, \dots, 6),$$

via the following birational and symplectic transformations

$$\begin{aligned} 1) x_1 &= \frac{1}{x}, & y_1 &= -(yx + \alpha_1)x, & z_1 &= z, & w_1 &= w, & q_1 &= q, & p_1 &= p, \\ 2) x_2 &= x, & y_2 &= y, & z_2 &= \frac{1}{z}, & w_2 &= -(zw + \alpha_2)z, & q_2 &= q, & p_2 &= p, \\ 3) x_3 &= x, & y_3 &= y, & z_3 &= z, & w_3 &= w, & q_3 &= \frac{1}{q}, & p_3 &= -(qp + \alpha_3)q, \\ 4) x_4 &= -(yx + wz + pq - \alpha_4)y, & y_4 &= \frac{1}{y}, & z_4 &= zy, \\ & w_4 &= \frac{w}{y}, & q_4 &= qy, & p_4 &= \frac{p}{y}, \\ 5) x_5 &= -((x-1)y + (z-1)w + (q-1)p - \alpha_5)y, & y_5 &= \frac{1}{y}, & z_5 &= (z-1)y, \\ & w_5 &= \frac{w}{y}, & q_5 &= (q-1)y, & p_5 &= \frac{p}{y}, \\ 6) x_6 &= \frac{1}{x}, & y_6 &= -((y + tw/s + tp/u + t)x + \alpha_6)x, & z_6 &= z - tx/s, \\ & w_6 &= w, & q_6 &= q - tx/u, & p_6 &= p. \end{aligned}$$

Then such a system coincides with the system (31).

Theorem 6.2. *The system (31) is invariant under the following transformations: with the nota-*

tion $(*) = (x, y, z, w, q, p, t, s, u; \alpha_1, \alpha_2, \dots, \alpha_6)$,

$$\begin{aligned}
s_1 : (*) &\rightarrow \left(x + \frac{\alpha_1}{y}, y, z, w, q, p, t, s, u; -\alpha_1, \alpha_2, \alpha_3, \alpha_4 + \alpha_1, \alpha_5 + \alpha_1, \alpha_6 \right), \\
s_2 : (*) &\rightarrow \left(x, y, z + \frac{\alpha_2}{w}, w, t, s, u; \alpha_1, -\alpha_2, \alpha_3, \alpha_4 + \alpha_2, \alpha_5 + \alpha_2, \alpha_6 \right), \\
s_3 : (*) &\rightarrow \left(x, y, z, w, q + \frac{\alpha_3}{p}, p, t, s, u; \alpha_1, \alpha_2, -\alpha_3, \alpha_4 + \alpha_3, \alpha_5 + \alpha_3, \alpha_6 \right), \\
s_6 : (*) &\rightarrow \left(x + \frac{\alpha_6 s u}{s u y + t u w + t s p + t s u}, y, z + \frac{\alpha_6 t u}{s u y + t u w + t s p + t s u}, w, \right. \\
&\quad \left. q + \frac{\alpha_6 t s}{s u y + t u w + t s p + t s u}, p, t, s, u; \alpha_1, \alpha_2, \alpha_3, \alpha_4 + \alpha_6, \alpha_5 + \alpha_6, -\alpha_6 \right), \\
\pi_1 : (*) &\rightarrow (z, w, q, p, x, y, s, u, t; \alpha_3, \alpha_2, \alpha_1, \alpha_4, \alpha_5, \alpha_6), \\
\pi_2 : (*) &\rightarrow (1 - x, -y, 1 - z, -w, 1 - q, -p, -t, -s, -u; \alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_4, \alpha_6), \\
\pi_3 : (*) &\rightarrow \left(-\frac{x y + z w + q p - \alpha_4}{t x}, \frac{t x (x y + \alpha_1)}{x y + z w + q p - \alpha_4}, -\frac{x y + z w + q p - \alpha_4}{s z}, \right. \\
&\quad \frac{s z (z w + \alpha_2)}{x y + z w + q p - \alpha_4}, -\frac{x y + z w + q p - \alpha_4}{u q}, \frac{u q (q p + \alpha_3)}{x y + z w + q p - \alpha_4}, -t, -s, -u; \\
&\quad \alpha_1, \alpha_2, \alpha_3, \alpha_5 + \alpha_6 - 1, \alpha_4 + \alpha_6, 1 - \alpha_6 \right), \\
\pi_4 : (*) &\rightarrow \left(\frac{(1 - x)y + (1 - z)w + (1 - q)p + \alpha_5}{t(x - 1)}, \right. \\
&\quad \frac{t(x - 1)((x - 1)y + \alpha_1)}{(x - 1)y + (z - 1)w + (q - 1)p - \alpha_5}, \frac{(1 - x)y + (1 - z)w + (1 - q)p + \alpha_5}{s(z - 1)}, \\
&\quad \frac{s(z - 1)((z - 1)w + \alpha_2)}{(x - 1)y + (z - 1)w + (q - 1)p - \alpha_5}, \frac{(1 - x)y + (1 - z)w + (1 - q)p + \alpha_5}{u(q - 1)}, \\
&\quad \left. \frac{u(q - 1)((q - 1)p + \alpha_3)}{(x - 1)y + (z - 1)w + (q - 1)p - \alpha_5}, t, s, u; \alpha_1, \alpha_2, \alpha_3, \alpha_4 + \alpha_6 - 1, \alpha_5 + \alpha_6, 1 - \alpha_6 \right).
\end{aligned}$$

7 Degeneration from the system (25)

As the fourth-order analogue of the confluence process from P_V to P_{IV} (see [7]), we consider the following coupling confluence process from the system (25). We take the following coupling confluence process $P_V \rightarrow P_{IV}$ for each coordinate system (x, y) and (z, w) of the system (25).

Theorem 7.1. *For the system (25), we make the change of parameters and variables*

$$\alpha_1 = A_1, \quad \alpha_2 = A_2, \quad \alpha_3 = A_3, \quad \alpha_4 = -\frac{1}{2\varepsilon^2}, \quad \alpha_5 = 1 - A_1 - A_2 - A_3 + \frac{1}{2\varepsilon^2} \quad (34)$$

$$\begin{aligned}
t &= \frac{1 + 2\varepsilon T}{2\varepsilon^2}, \quad s = \frac{1 + 2\varepsilon S}{2\varepsilon^2}, \quad x = \frac{\varepsilon X}{\varepsilon X - 1}, \quad z = \frac{\varepsilon Z}{\varepsilon Z - 1}, \\
y &= -\frac{(\varepsilon X - 1)\{(\varepsilon X - 1)Y + \varepsilon A_2\}}{\varepsilon}, \quad w = -\frac{(\varepsilon Z - 1)\{(\varepsilon Z - 1)W + \varepsilon A_2\}}{\varepsilon}
\end{aligned} \quad (35)$$

from $\alpha_1, \alpha_2, \dots, \alpha_5, t, x, y, z, w$ to $A_1, \dots, A_4, \varepsilon, T, X, Y, Z, W$. Then this system can also be written in the new variables T, X, Y, Z, W and parameters $A_1, A_2, A_3, A_4, \varepsilon$ as a Hamiltonian system. This new system tends to the polynomial Hamiltonian system

$$\begin{aligned}
dx &= \frac{\partial H_1}{\partial y} dt + \frac{\partial H_2}{\partial y} ds, & dy &= -\frac{\partial H_1}{\partial x} dt - \frac{\partial H_2}{\partial x} ds, \\
dz &= \frac{\partial H_1}{\partial w} dt + \frac{\partial H_2}{\partial w} ds, & dw &= -\frac{\partial H_1}{\partial z} dt - \frac{\partial H_2}{\partial z} ds
\end{aligned} \quad (36)$$

with the symmetric Hamiltonians $H_i \in \mathbb{C}(t, s)[x, y, z, w]$ ($i = 1, 2$)

$$\begin{aligned} H_1 &= H_{IV}(x, y, t; \alpha_1, \alpha_2) + \frac{\alpha_3}{t-s}xy + \frac{\alpha_2}{t-s}zw \\ &\quad - \frac{x^2yw - 2(t-s)yzw - 2xyzw + yz^2w + \alpha_3yz + \alpha_2xw}{t-s}, \\ H_2 &= \pi(H_1) \quad (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1) \end{aligned} \quad (37)$$

as $\varepsilon \rightarrow 0$.

Here, for notational convenience, we have renamed $X, Y, Z, W, T, S, A_1, A_2, A_3$ to $x, y, z, w, t, s, \alpha_1, \alpha_2, \alpha_3$ (which are not the same as the previous $x, y, z, w, t, s, \alpha_1, \alpha_2, \alpha_3$). The transformation π is explicitly given by

$$\pi : (x, y, z, w, t, s; \alpha_1, \alpha_2, \alpha_3, \alpha_4) \rightarrow (z, w, x, y, s, t; \alpha_1, \alpha_3, \alpha_2, \alpha_4), \quad (38)$$

and the symbol $H_{IV}(x, y, t; \alpha_1, \alpha_2)$ denotes the fourth Painlevé Hamiltonian given by

$$H_{IV}(x, y, t; \alpha_1, \alpha_2) = -x^2y + 2xy^2 - 2txy - 2\alpha_1y - \alpha_2x \quad (\alpha_0 + \alpha_1 + \alpha_2 = 1). \quad (39)$$

Theorem 7.2. *Let us consider a polynomial Hamiltonian system with Hamiltonian $H_i \in \mathbb{C}(t, s)[x, y, z, w]$ ($i = 1, 2$). We assume that*

(A1) *$\deg(H_i) = 5$ with respect to x, y, z, w .*

(A2) *This system becomes again a polynomial Hamiltonian system in each coordinate system $\{U_j, (x_j, y_j, z_j, w_j)\}$ ($j = 1, 2, 3, 4$):*

$$U_j = \mathbb{C}^4 \ni (x_j, y_j, z_j, w_j) \quad (j = 1, 2, 3, 4),$$

via the following birational and symplectic transformations

$$\begin{aligned} 1) x_1 &= -(xy + zw - \alpha_1)y, \quad y_1 = \frac{1}{y}, \quad z_1 = zy, \quad w_1 = \frac{w}{y}, \\ 2) x_2 &= \frac{1}{x}, \quad y_2 = -(yx + \alpha_2)x, \quad z_2 = z, \quad w_2 = w, \\ 3) x_3 &= x, \quad y_3 = y, \quad z_3 = \frac{1}{z}, \quad w_3 = -(zw + \alpha_3)z, \\ 4) x_4 &= -((x - 2y - 2w + 2t)y + (z - 2y - 2w + 2s)w - \alpha_4)y, \quad y_4 = \frac{1}{y}, \\ z_4 &= (z - 2y - 2w + 2s)y, \quad w_4 = \frac{w}{y}. \end{aligned}$$

Then such a system coincides with the system (36).

Theorem 7.3. *The system (36) is invariant under the following transformations: with the notation $(*) = (x, y, z, w, t, s; \alpha_1, \alpha_2, \alpha_3, \alpha_4)$,*

$$\begin{aligned} s_2 : (*) &\rightarrow \left(x + \frac{\alpha_2}{y}, y, z, w, t, s; \alpha_1 + \alpha_2, -\alpha_2, \alpha_3, \alpha_4 + \alpha_2 \right), \\ s_3 : (*) &\rightarrow \left(x, y, z + \frac{\alpha_3}{w}, w, t, s; \alpha_1 + \alpha_3, \alpha_2, -\alpha_3, \alpha_4 + \alpha_3 \right), \\ \pi_1 : (*) &\rightarrow (z, w, x, y, s, t; \alpha_1, \alpha_3, \alpha_2, \alpha_4), \\ \pi_2 : (*) &\rightarrow (\sqrt{-1}(x - 2y - 2w + 2t), -\sqrt{-1}y, \sqrt{-1}(z - 2y - 2w + 2s), -\sqrt{-1}w, \\ &\quad -\sqrt{-1}t, -\sqrt{-1}s; \alpha_4, \alpha_2, \alpha_3, \alpha_1), \\ \pi_3 : (*) &\rightarrow \left(\frac{2\sqrt{-1}(xy + zw - \alpha_1)}{x}, \frac{\sqrt{-1}x(xy + \alpha_2)}{2(xy + zw - \alpha_1)}, \frac{2\sqrt{-1}(xy + zw - \alpha_1)}{z}, \right. \\ &\quad \left. \frac{\sqrt{-1}z(zw + \alpha_3)}{2(xy + zw - \alpha_1)}, -\sqrt{-1}t, -\sqrt{-1}s; -\alpha_1 - \alpha_2 - \alpha_3, \alpha_2, \alpha_3, 1 + \alpha_1 \right). \end{aligned}$$

8 A generalization of the system (36) to three variables

In this section, we present a generalization of the system (36) to three variables, which is equivalent to the polynomial Hamiltonian system

$$\begin{aligned} dx &= \frac{\partial H_1}{\partial y} dt + \frac{\partial H_2}{\partial y} ds + \frac{\partial H_3}{\partial y} du, & dy &= -\frac{\partial H_1}{\partial x} dt - \frac{\partial H_2}{\partial x} ds - \frac{\partial H_3}{\partial x} du, \\ dz &= \frac{\partial H_1}{\partial w} dt + \frac{\partial H_2}{\partial w} ds + \frac{\partial H_3}{\partial w} du, & dw &= -\frac{\partial H_1}{\partial z} dt - \frac{\partial H_2}{\partial z} ds - \frac{\partial H_3}{\partial z} du, \\ dq &= \frac{\partial H_1}{\partial p} dt + \frac{\partial H_2}{\partial p} ds + \frac{\partial H_3}{\partial p} du, & dp &= -\frac{\partial H_1}{\partial q} dt - \frac{\partial H_2}{\partial q} ds - \frac{\partial H_3}{\partial q} du \end{aligned} \quad (40)$$

with the symmetric Hamiltonians $H_i \in \mathbb{C}(t, s, u)[x, y, z, w, q, p]$ ($i = 1, 2, 3$)

$$\begin{aligned} H_1 &= H_{IV}(x, y, t; \alpha_1, \alpha_2) \\ &\quad + R(x, y, z, w, t, s; \alpha_2, \alpha_3) + R(x, y, q, p, t, u; \alpha_2, \alpha_4), \\ H_2 &= \pi(H_1), \quad H_3 = (\pi \circ \pi)(H_1) \quad (\alpha_1 + \alpha_2 + \dots + \alpha_5 = 1), \end{aligned} \quad (41)$$

where the transformation π is explicitly given by

$$\pi : (*) \rightarrow (z, w, q, p, x, y, s, u, t; \alpha_1, \alpha_3, \alpha_4, \alpha_2, \alpha_5). \quad (42)$$

Here, the symbol $(*)$ denotes $(*) := (x, y, z, w, q, p, t, s, u; \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$, and the symbol $R(q_l, p_l, q_m, p_m, t_l, t_m; \alpha, \beta)$ is explicitly given by

$$\begin{aligned} &R(q_l, p_l, q_m, p_m, t_l, t_m; \alpha, \beta) \\ &= \frac{\beta q_l p_l}{t_l - t_m} + \frac{\alpha q_m p_m}{t_l - t_m} \\ &\quad - \frac{q_l^2 p_l p_m + 2t_m p_l q_m p_m - 2t_l p_l q_m p_m - 2q_l p_l q_m p_m + p_l q_m^2 p_m + \beta p_l q_m + \alpha q_l p_m}{t_l - t_m}. \end{aligned}$$

Theorem 8.1. *Let us consider a polynomial Hamiltonian system with Hamiltonian $H_i \in \mathbb{C}(t, s, u)[x, y, z, w, q, p]$ ($i = 1, 2, 3$). We assume that*

(A1) *$\deg(H_i) = 5$ with respect to x, y, z, w, q, p .*

(A2) *This system becomes again a polynomial Hamiltonian system in each coordinate system $\{U_j, (x_j, y_j, z_j, w_j, q_j, p_j)\}$ ($j = 1, 2, \dots, 5$):*

$$U_j = \mathbb{C}^6 \ni (x_j, y_j, z_j, w_j, q_j, p_j) \quad (j = 1, 2, \dots, 5),$$

via the following birational and symplectic transformations

$$\begin{aligned} 1) & x_1 = -(yx + wz + pq - \alpha_1)y, \quad y_1 = \frac{1}{y}, \quad z_1 = zy, \\ & w_1 = \frac{w}{y}, \quad q_1 = qy, \quad p_1 = \frac{p}{y}, \\ 2) & x_2 = \frac{1}{x}, \quad y_2 = -(yx + \alpha_2)x, \quad z_2 = z, \quad w_2 = w, \quad q_2 = q, \quad p_2 = p, \\ 3) & x_3 = x, \quad y_3 = y, \quad z_3 = \frac{1}{z}, \quad w_3 = -(wz + \alpha_3)z, \quad q_3 = q, \quad p_3 = p, \\ 4) & x_4 = x, \quad y_4 = y, \quad z_4 = z, \quad w_4 = w, \quad q_4 = \frac{1}{q}, \quad p_4 = -(qp + \alpha_4)q, \\ 5) & x_5 = -\{(x - 2y - 2w - 2p + 2t)y + (z - 2y - 2w - 2p + 2s)w \\ & \quad + (q - 2y - 2w - 2p + 2u)p - \alpha_5\}y, \quad y_5 = \frac{1}{y}, \quad z_5 = (z - 2y - 2w - 2p + 2s)y, \\ & w_5 = \frac{w}{y}, \quad q_5 = (q - 2y - 2w - 2p + 2u)y, \quad p_5 = \frac{p}{y}. \end{aligned}$$

Then such a system coincides with the system (40).

Theorem 8.2. *The system (40) is invariant under the following transformations: with the notation $(*) = (x, y, z, w, q, p, t, s, u; \alpha_1, \alpha_2, \dots, \alpha_5)$,*

$$\begin{aligned}
s_2 : (*) &\rightarrow \left(x + \frac{\alpha_2}{y}, y, z, w, q, p, t, s, u; \alpha_1 + \alpha_1, -\alpha_2, \alpha_3, \alpha_4, \alpha_5 + \alpha_2 \right), \\
s_3 : (*) &\rightarrow \left(x, y, z + \frac{\alpha_3}{w}, w, t, s, u; \alpha_1 + \alpha_3, \alpha_2, -\alpha_3, \alpha_4, \alpha_5 + \alpha_3 \right), \\
s_4 : (*) &\rightarrow \left(x, y, z, w, q + \frac{\alpha_4}{p}, p, t, s, u; \alpha_1 + \alpha_4, \alpha_2, \alpha_3, -\alpha_4, \alpha_5 + \alpha_4 \right), \\
\pi_1 : (*) &\rightarrow (z, w, q, p, x, y, s, u, t; \alpha_1, \alpha_3, \alpha_4, \alpha_2, \alpha_5), \\
\pi_2 : (*) &\rightarrow (\sqrt{-1}(x - 2y - 2w - 2p + 2t), -\sqrt{-1}y, \sqrt{-1}(z - 2y - 2w - 2p + 2s), \\
&\quad -\sqrt{-1}w, \sqrt{-1}(q - 2y - 2w - 2p + 2u), -\sqrt{-1}p, -\sqrt{-1}t, -\sqrt{-1}s, \\
&\quad -\sqrt{-1}u; \alpha_5, \alpha_2, \alpha_3, \alpha_4, \alpha_1), \\
\pi_3 : (*) &\rightarrow \left(\frac{2\sqrt{-1}(xy + zw + qp - \alpha_1)}{x}, \frac{\sqrt{-1}x(xy + \alpha_2)}{2(xy + zw + qp - \alpha_1)}, \right. \\
&\quad \frac{2\sqrt{-1}(xy + zw + qp - \alpha_1)}{z}, \frac{\sqrt{-1}z(zw + \alpha_3)}{2(xy + zw + qp - \alpha_1)}, \\
&\quad \frac{2\sqrt{-1}(xy + zw + qp - \alpha_1)}{q}, \frac{\sqrt{-1}q(qp + \alpha_4)}{2(xy + zw + qp - \alpha_1)}, \\
&\quad \left. -\sqrt{-1}t, -\sqrt{-1}s, -\sqrt{-1}u; -\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4, \alpha_2, \alpha_3, \alpha_4, 1 + \alpha_1 \right).
\end{aligned}$$

9 Another degeneration from the system (25)

As the fourth-order analogue of the above confluence process from P_V to P_{III} (see [7]), we consider the following coupling confluence process from the system (25). We take the following coupling confluence process $P_V \rightarrow P_{III}$ for each coordinate system (x, y) and (z, w) of the system (25).

Theorem 9.1. *For the system (25), we make the change of parameters and variables*

$$\alpha_1 = A_0, \quad \alpha_2 = A_2, \quad \alpha_3 = \frac{1}{\varepsilon}, \quad \alpha_4 = 2A_1 - \frac{1}{\varepsilon}, \quad \alpha_5 = A_3 \quad (43)$$

$$\begin{aligned}
t &= -\varepsilon T, \quad s = -\varepsilon S, \quad X = -t(x - 1), \quad Z = -s(z - 1), \\
Y &= -\frac{y}{t}, \quad W = -\frac{w}{s}
\end{aligned} \quad (44)$$

from $\alpha_1, \alpha_2, \dots, \alpha_5, t, s, x, y, z, w$ to $A_0, \dots, A_3, \varepsilon, T, S, X, Y, Z, W$. Then this system can also be written in the new variables T, S, X, Y, Z, W and parameters $A_0, \dots, A_3, \varepsilon$ as a Hamiltonian system. This new system tends to the polynomial Hamiltonian system

$$\begin{aligned}
dx &= \frac{\partial H_1}{\partial y} dt + \frac{\partial H_2}{\partial y} ds, \quad dy = -\frac{\partial H_1}{\partial x} dt - \frac{\partial H_2}{\partial x} ds, \\
dz &= \frac{\partial H_1}{\partial w} dt + \frac{\partial H_2}{\partial w} ds, \quad dw = -\frac{\partial H_1}{\partial z} dt - \frac{\partial H_2}{\partial z} ds
\end{aligned} \quad (45)$$

with the symmetric Hamiltonians $H_i \in \mathbb{C}(t, s)[x, y, z, w]$ ($i = 1, 2$)

$$\begin{aligned}
H_1 &= H_{III}(x, y, t; \alpha_0, \alpha_1) + \frac{\alpha_2 s}{t(t-s)} xy + \frac{\alpha_0}{t-s} zw \\
&\quad - \frac{tyz^2w + sx^2yw - 2txyzw + \alpha_2 tyz + \alpha_0 sxw}{t(t-s)}, \\
H_2 &= \pi(H_1) \quad (\alpha_0 + 2\alpha_1 + \alpha_2 + \alpha_3 = 1),
\end{aligned} \quad (46)$$

as $\varepsilon \rightarrow 0$.

Here, for notational convenience, we have renamed $X, Y, Z, W, T, S, A_1, A_2, A_3$ to $x, y, z, w, t, s, \alpha_1, \alpha_2, \alpha_3$ (which are not the same as the previous $x, y, z, w, t, s, \alpha_1, \alpha_2, \alpha_3$). The transformation π is explicitly given by

$$\pi : (x, y, z, w, t, s; \alpha_0, \alpha_1, \alpha_2, \alpha_3) \rightarrow (z, w, x, y, s, t; \alpha_2, \alpha_1, \alpha_0, \alpha_3), \quad (47)$$

and the symbol $H_{III}(x, y, t; \alpha_0, \alpha_1, \alpha_2)$ denotes the third Painlevé Hamiltonian given by

$$H_{III}(x, y, t; \alpha_0, \alpha_1) = \frac{x^2 y(y-1) + x\{(1-2\alpha_1)y - \alpha_0\} + ty}{t} \quad (\alpha_0 + 2\alpha_1 + \alpha_2 = 1). \quad (48)$$

Theorem 9.2. *Let us consider a polynomial Hamiltonian system with Hamiltonian*

$H_i \in \mathbb{C}(t, s)[x, y, z, w]$ ($i = 1, 2$). *We assume that*

(A1) $\deg(H_i) = 5$ *with respect to* x, y, z, w .

(A2) *This system becomes again a polynomial Hamiltonian system in each coordinate system* $\{U_j, (x_j, y_j, z_j, w_j)\}$ ($j = 0, 1, 2, 3$):

$$U_j = \mathbb{C}^4 \ni (x_j, y_j, z_j, w_j) \quad (j = 0, 1, 2, 3),$$

via the following birational and symplectic transformations

$$\begin{aligned} 0) x_0 &= \frac{1}{x}, \quad y_0 = -(yx + \alpha_0)x, \quad z_0 = z, \quad w_0 = w, \\ 1) x_1 &= x, \quad y_1 = y + \frac{s}{t}w + \frac{2\{(z - \frac{s}{t}x)w - \alpha_1\}}{x} + \frac{t}{x^2}, \\ z_1 &= \frac{z - \frac{s}{t}x}{x^2}, \quad w_1 = x^2 w, \\ 2) x_2 &= x, \quad y_2 = y, \quad z_2 = \frac{1}{z}, \quad w_2 = -(zw + \alpha_2)z, \\ 3) x_3 &= \frac{1}{x}, \quad y_3 = -((y + w - 1)x + \alpha_3)x, \quad z_3 = z - x, \quad w_3 = w. \end{aligned}$$

Then such a system coincides with the system (45).

Theorem 9.3. *The system (45) is invariant under the following transformations: with the notation $(*) = (x, y, z, w, t, s; \alpha_1, \alpha_2, \alpha_3, \alpha_4)$,*

$$\begin{aligned} s_0 : (*) &\rightarrow \left(x + \frac{\alpha_0}{y}, y, z, w, t, s; -\alpha_0, \alpha_1 + \alpha_0, \alpha_2, \alpha_3\right), \\ s_2 : (*) &\rightarrow \left(x, y, z + \frac{\alpha_2}{w}, w, t, s; \alpha_0, \alpha_1 + \alpha_2, -\alpha_2, \alpha_3\right), \\ s_3 : (*) &\rightarrow \left(x + \frac{\alpha_3}{y + w - 1}, y, z + \frac{\alpha_3}{y + w - 1}, w, t, s; \alpha_0, \alpha_1 + \alpha_3, \alpha_2, -\alpha_3\right), \\ \pi_1 : (*) &\rightarrow (z, w, x, y, s, t; \alpha_2, \alpha_1, \alpha_0, \alpha_3), \\ \pi_2 : (*) &\rightarrow \left(\frac{t}{x}, -\frac{(xy + \alpha_0)x}{t}, \frac{s}{z}, -\frac{(zw + \alpha_2)z}{s}, t, s; \alpha_0, \alpha_1 + \alpha_3 - \frac{1}{2}, \alpha_2, 1 - \alpha_3\right), \\ \pi_3 : (*) &\rightarrow (x - z, y, -z, 1 - y - w, t - s, -s; \alpha_0, \alpha_1, \alpha_3, \alpha_2). \end{aligned}$$

10 A generalization of the system (45) to three variables

In this section, we present a generalization of the system (45) to three variables, which is equivalent to the polynomial Hamiltonian system

$$\begin{aligned} dx &= \frac{\partial H_1}{\partial y} dt + \frac{\partial H_2}{\partial y} ds + \frac{\partial H_3}{\partial y} du, & dy &= -\frac{\partial H_1}{\partial x} dt - \frac{\partial H_2}{\partial x} ds - \frac{\partial H_3}{\partial x} du, \\ dz &= \frac{\partial H_1}{\partial w} dt + \frac{\partial H_2}{\partial w} ds + \frac{\partial H_3}{\partial w} du, & dw &= -\frac{\partial H_1}{\partial z} dt - \frac{\partial H_2}{\partial z} ds - \frac{\partial H_3}{\partial z} du, \\ dq &= \frac{\partial H_1}{\partial p} dt + \frac{\partial H_2}{\partial p} ds + \frac{\partial H_3}{\partial p} du, & dp &= -\frac{\partial H_1}{\partial q} dt - \frac{\partial H_2}{\partial q} ds - \frac{\partial H_3}{\partial q} du \end{aligned} \quad (49)$$

with the symmetric Hamiltonians $H_i \in \mathbb{C}(t, s, u)[x, y, z, w, q, p]$ ($i = 1, 2, 3$)

$$\begin{aligned} H_1 &= H_{III}(x, y, t; \alpha_0, \alpha_1) \\ &\quad + R(x, y, z, w, t, s; \alpha_0, \alpha_2) + R(x, y, q, p, t, u; \alpha_0, \alpha_4), \\ H_2 &= \pi(H_1), \quad H_3 = (\pi \circ \pi)(H_1) \quad (\alpha_0 + 2\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1), \end{aligned} \quad (50)$$

where the transformation π is explicitly given by

$$\pi : (*) \rightarrow (z, w, q, p, x, y, s, u, t; \alpha_2, \alpha_1, \alpha_4, \alpha_3, \alpha_0). \quad (51)$$

Here the symbol $(*)$ denotes $(*) := (x, y, z, w, q, p, t, s, u; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$, and the symbol $R(q_l, p_l, q_m, p_m, t_l, t_m; \alpha, \beta)$ is explicitly given by

$$\begin{aligned} &R(q_l, p_l, q_m, p_m, t_l, t_m; \alpha, \beta) \\ &= \frac{\beta t_m q_l p_l}{t_l(t_l - t_m)} + \frac{\alpha q_m p_m}{t_l - t_m} \\ &\quad - \frac{t_l p_l q_m^2 p_m + t_m q_l^2 p_l p_m - 2t_l q_l p_l q_m p_m + \beta t_l p_l q_m + \alpha t_m q_l p_m}{t_l(t_l - t_m)}. \end{aligned}$$

Theorem 10.1. *Let us consider a polynomial Hamiltonian system with Hamiltonian $H_i \in \mathbb{C}(t, s, u)[x, y, z, w, q, p]$ ($i = 1, 2, 3$). We assume that*

(A1) *$\deg(H_i) = 5$ with respect to x, y, z, w, q, p .*

(A2) *This system becomes again a polynomial Hamiltonian system in each coordinate system $\{U_j, (x_j, y_j, z_j, w_j, q_j, p_j)\}$ ($j = 0, 1, \dots, 4$):*

$$U_j = \mathbb{C}^6 \ni (x_j, y_j, z_j, w_j, q_j, p_j) \quad (j = 0, 1, \dots, 4),$$

via the following birational and symplectic transformations

$$\begin{aligned} 0) & x_0 = \frac{1}{x}, \quad y_0 = -(yx + \alpha_0)x, \quad z_0 = z, \quad w_0 = w, \quad q_0 = q, \quad p_0 = p, \\ 1) & x_1 = x, \quad y_1 = y + \frac{s}{t}w + \frac{u}{t}p + \frac{2\{(z - \frac{s}{t}x)w + (q - \frac{u}{t}x)p - \alpha_1\}}{x} + \frac{t}{x^2}, \\ & z_1 = \frac{z - \frac{s}{t}x}{x^2}, \quad w_1 = x^2w, \quad q_1 = \frac{q - \frac{u}{t}x}{x^2}, \quad p_1 = x^2p, \\ 2) & x_2 = x, \quad y_2 = y, \quad z_2 = \frac{1}{z}, \quad w_2 = -(zw + \alpha_2)z, \quad q_2 = q, \quad p_2 = p, \\ 3) & x_3 = \frac{1}{x}, \quad y_3 = -((y + w + p - 1)x + \alpha_3)x, \quad z_3 = z - x, \\ & w_3 = w, \quad q_3 = q - x, \quad p_3 = p, \\ 4) & x_4 = x, \quad y_4 = y, \quad z_4 = z, \quad w_4 = w, \quad q_4 = \frac{1}{q}, \quad p_4 = -(qp + \alpha_4)q. \end{aligned}$$

Then such a system coincides with the system (50).

Theorem 10.2. *The system (50) is invariant under the following transformations: with the*

notation $(*) = (x, y, z, w, q, p, t, s, u; \alpha_0, \alpha_1, \dots, \alpha_4)$,

$$\begin{aligned}
s_0 : (*) &\rightarrow \left(x + \frac{\alpha_0}{y}, y, z, w, q, p, t, s, u; -\alpha_0, \alpha_1 + \alpha_0, \alpha_2, \alpha_3, \alpha_4 \right), \\
s_2 : (*) &\rightarrow \left(x, y, z + \frac{\alpha_2}{w}, w, q, p, t, s, u; \alpha_0, \alpha_1 + \alpha_2, -\alpha_2, \alpha_3, \alpha_4 \right), \\
s_3 : (*) &\rightarrow \left(x + \frac{\alpha_3}{y + w + p - 1}, y, z + \frac{\alpha_3}{y + w + p - 1}, w, q + \frac{\alpha_3}{y + w + p - 1}, p, \right. \\
&\quad \left. t, s, u; \alpha_0, \alpha_1 + \alpha_3, \alpha_2, -\alpha_3, \alpha_4 \right), \\
s_4 : (*) &\rightarrow \left(x, y, z, w, q + \frac{\alpha_4}{p}, p, t, s, u; \alpha_0, \alpha_1 + \alpha_4, \alpha_2, \alpha_3, -\alpha_4 \right), \\
\pi_1 : (*) &\rightarrow (z, w, q, p, x, y, s, u, t; \alpha_2, \alpha_1, \alpha_4, \alpha_3, \alpha_0), \\
\pi_2 : (*) &\rightarrow \left(\frac{t}{x}, -\frac{(xy + \alpha_0)x}{t}, \frac{s}{z}, -\frac{(zw + \alpha_2)z}{s}, \frac{u}{q}, -\frac{(qp + \alpha_4)q}{u}, \right. \\
&\quad \left. t, s, u; \alpha_0, \alpha_1 + \alpha_3 - \frac{1}{2}, \alpha_2, 1 - \alpha_3, \alpha_4 \right).
\end{aligned}$$

11 Other generalization of the third Painlevé system

In this section, we find a generalization of the third Painlevé system to two variables t, s , which is different from the system (45). This system is equivalent to the Hamiltonian system

$$\begin{aligned}
dx &= \frac{\partial H_1}{\partial y} dt + \frac{\partial H_2}{\partial y} ds, & dy &= -\frac{\partial H_1}{\partial x} dt - \frac{\partial H_2}{\partial x} ds, \\
dz &= \frac{\partial H_1}{\partial w} dt + \frac{\partial H_2}{\partial w} ds, & dw &= -\frac{\partial H_1}{\partial z} dt - \frac{\partial H_2}{\partial z} ds
\end{aligned} \tag{52}$$

with the polynomial Hamiltonians $H_i \in \mathbb{C}(t, s)[x, y, z, w]$ ($i = 1, 2$)

$$\begin{aligned}
H_1 &= \frac{-x^3 y^2 + s x^2 y^2 - (2\alpha_1 + \alpha_2) x^2 y + \{(2\alpha_1 + \alpha_2)s + \eta_1 t\}xy}{ts} - \frac{\alpha_1(\alpha_1 + \alpha_2)x + \eta_1 t s y}{ts} \\
&+ \frac{z^3 w^2 - t z^2 w^2 + (2\alpha_1 + \alpha_2) z^2 w + (\alpha_3 t - \eta_0 s) z w}{t^2} + \frac{\alpha_1(\alpha_1 + \alpha_2)z + \eta_0 t s w}{t^2} \\
&- \frac{\eta_0 s - \alpha_3 t}{t^2} xy + \frac{\eta_1}{s} z w - \frac{xz\{2(tx - sz)yw - sxy^2 + t z w^2 - (2\alpha_1 + \alpha_2)(sy - tw)\}}{t^2 s}, \\
H_2 &= \pi(H_1) \quad (2\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1),
\end{aligned} \tag{53}$$

where the transformation π is explicitly given by

$$\pi : (x, y, z, w, t, s; \eta_0, \eta_1, \alpha_1, \alpha_2, \alpha_3, \alpha_4) \rightarrow (z, w, x, y, s, t; \eta_1, \eta_0, \alpha_1, \alpha_2, \alpha_4, \alpha_3). \tag{54}$$

Theorem 11.1. *Let us consider a polynomial Hamiltonian system with Hamiltonian $H_i \in \mathbb{C}(t, s)[x, y, z, w]$ ($i = 1, 2$). We assume that*

(A1) *$\deg(H_i) = 5$ with respect to x, y, z, w .*

(A2) *This system becomes again a polynomial Hamiltonian system in each coordinate system $\{U_j, (x_j, y_j, z_j, w_j)\}$ ($j = 1, 2, 3, 4$):*

$$U_j = \mathbb{C}^4 \ni (x_j, y_j, z_j, w_j) \quad (j = 1, 2, 3, 4),$$

via the following birational and symplectic transformations

$$\begin{aligned}
1) x_1 &= \frac{1}{x}, & y_1 &= -(xy + zw + \alpha_1)x, & z_1 &= \frac{z}{x}, & w_1 &= xw, \\
2) x_2 &= \frac{1}{x}, & y_2 &= -(xy + zw + \alpha_1 + \alpha_2)x, & z_2 &= \frac{z}{x}, & w_2 &= xw, \\
3) x_3 &= x, & y_3 &= y - \frac{\eta_0}{z}, & z_3 &= z, & w_3 &= w - \frac{\alpha_3}{z} + \frac{\eta_0(x - s)}{z^2}, \\
4) x_4 &= x, & y_4 &= y - \frac{\alpha_4}{x} + \frac{\eta_1(z - t)}{x^2}, & z_4 &= z, & w_4 &= w - \frac{\eta_1}{x}.
\end{aligned}$$

Then such a system coincides with the system (52).

Theorem 11.2. *The system (52) is invariant under the following transformations: with the notation $(*) = (x, y, z, w, t, s; \eta_0, \eta_1, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$,*

$$\begin{aligned} s_2 : (*) &\rightarrow (x, y, z, w, t, s; \eta_0, \eta_1, \alpha_1 + \alpha_2, -\alpha_2, \alpha_3, \alpha_4), \\ s_3 : (*) &\rightarrow \left(x, y - \frac{\eta_0}{z}, z, w - \frac{\alpha_3}{z} + \frac{\eta_0(x-s)}{z^2}, t, s; -\eta_0, \eta_1, \alpha_1 + \alpha_3, \alpha_2, -\alpha_3, \alpha_4 \right), \\ s_4 : (*) &\rightarrow \left(x, y - \frac{\alpha_4}{x} + \frac{\eta_1(z-t)}{x^2}, z, w - \frac{\eta_1}{x}, t, s; \eta_0, -\eta_1, \alpha_1 + \alpha_4, \alpha_2, \alpha_3, -\alpha_4 \right). \end{aligned}$$

12 Degeneration from the system (36)

As the fourth-order analogue of the above confluence process from P_{IV} to P_{II} (see [7]), we consider the following coupling confluence process from the system (36). We take the following coupling confluence process $P_{IV} \rightarrow P_{II}$ for each coordinate system (x, y) and (z, w) of the system (36).

Theorem 12.1. *For the system (36), we make the change of parameters and variables*

$$\alpha_1 = \frac{1}{4\varepsilon^6}, \quad \alpha_2 = A_1, \quad \alpha_3 = A_3, \quad \alpha_4 = A_2 - \frac{1}{4\varepsilon^6}, \quad (55)$$

$$\begin{aligned} t &= -\frac{1 - \varepsilon^4 T}{\sqrt{2}\varepsilon^3}, \quad s = -\frac{1 - \varepsilon^4 S}{\sqrt{2}\varepsilon^3}, \quad x = \frac{1 + 2\varepsilon^2 X}{\sqrt{2}\varepsilon^3}, \\ z &= \frac{1 + 2\varepsilon^2 Z}{\sqrt{2}\varepsilon^3}, \quad y = \frac{\varepsilon Y}{\sqrt{2}}, \quad w = \frac{\varepsilon W}{\sqrt{2}} \end{aligned} \quad (56)$$

from $\alpha_1, \alpha_2, \dots, \alpha_4, t, s, x, y, z, w$ to $A_1, A_2, A_3, \varepsilon, T, S, X, Y, Z, W$. Then this system can also be written in the new variables T, S, X, Y, Z, W and parameters $A_1, A_2, A_3, \varepsilon$ as a Hamiltonian system. This new system tends to the polynomial Hamiltonian system

$$\begin{aligned} dx &= \frac{\partial H_1}{\partial y} dt + \frac{\partial H_2}{\partial y} ds, \quad dy = -\frac{\partial H_1}{\partial x} dt - \frac{\partial H_2}{\partial x} ds, \\ dz &= \frac{\partial H_1}{\partial w} dt + \frac{\partial H_2}{\partial w} ds, \quad dw = -\frac{\partial H_1}{\partial z} dt - \frac{\partial H_2}{\partial z} ds \end{aligned} \quad (57)$$

with the symmetric Hamiltonians $H_i \in \mathbb{C}(t, s)[x, y, z, w]$ ($i = 1, 2$)

$$H_1 = H_{II}(x, y, t; \alpha_3) + \frac{\alpha_1}{t-s}xy - \frac{\alpha_1}{t-s}yz - \frac{\alpha_3}{t-s}xw + \frac{\alpha_3}{t-s}zw - \frac{\{2(x-z)^2 - (t-s)\}yw}{2(t-s)}, \quad (58)$$

$$H_2 = \pi(H_1) \quad (\alpha_1 + \alpha_2 + \alpha_3 = 1)$$

as $\varepsilon \rightarrow 0$.

Here, for notational convenience, we have renamed $X, Y, Z, W, T, S, A_1, A_2$ to $x, y, z, w, t, s, \alpha_1, \alpha_2$ (which are not the same as the previous $x, y, z, w, t, s, \alpha_1, \alpha_2$). The transformation π is explicitly given by

$$\pi : (x, y, z, w, t, s; \alpha_1, \alpha_2, \alpha_3) \rightarrow (z, w, x, y, s, t; \alpha_3, \alpha_2, \alpha_1), \quad (59)$$

and the symbol $H_{II}(x, y, t; \alpha_1)$ denotes the second Painlevé Hamiltonian given by

$$H_{II}(x, y, t; \alpha_1) = \frac{1}{2}y^2 - \left(x^2 + \frac{t}{2}\right)y - \alpha_1 x. \quad (60)$$

Theorem 12.2. *Let us consider a polynomial Hamiltonian system with Hamiltonian $H_i \in \mathbb{C}(t, s)[x, y, z, w]$ ($i = 1, 2$). We assume that*

(A1) $\deg(H_i) = 5$ with respect to x, y, z, w .

(A2) This system becomes again a polynomial Hamiltonian system in each coordinate system $\{U_j, (x_j, y_j, z_j, w_j)\}$ ($j = 1, 2, 3$):

$$U_j = \mathbb{C}^4 \ni (x_j, y_j, z_j, w_j) \quad (j = 1, 2, 3),$$

via the following birational and symplectic transformations

$$\begin{aligned}
1) x_1 &= \frac{1}{x}, & y_1 &= -(yx + \alpha_3)x, & z_1 &= z, & w_1 &= w, \\
2) x_2 &= \frac{1}{x}, & y_2 &= -\{(y + w - 2x^2 - t)x - 2\left((z - x)x - \frac{t - s}{4}\right)\left(\frac{w}{x}\right) + \alpha_2\}x, \\
z_2 &= \left((z - x)x - \frac{t - s}{2}\right)x, & w_2 &= \frac{w}{x^2}, \\
3) x_3 &= x, & y_3 &= y, & z_3 &= \frac{1}{z}, & w_3 &= -(zw + \alpha_1)z.
\end{aligned}$$

Then such a system coincides with the system (57).

Theorem 12.3. *The system (57) is invariant under the following transformations: with the notation $(*) = (x, y, z, w, t, s; \alpha_1, \alpha_2, \alpha_3, \alpha_4)$,*

$$\begin{aligned}
s_1 : (*) &\rightarrow \left(x + \frac{\alpha_3}{y}, y, z, w, t, s; \alpha_1, \alpha_2 + \alpha_3, -\alpha_3\right), \\
s_3 : (*) &\rightarrow \left(x, y, z + \frac{\alpha_1}{w}, w, t, s; -\alpha_1, \alpha_2 + \alpha_1, \alpha_3\right), \\
\pi_1 : (*) &\rightarrow (z, w, x, y, s, t; \alpha_3, \alpha_2, \alpha_1).
\end{aligned}$$

For the system (57), we make the change of variables

$$t = T, \quad s = T + S \quad (61)$$

from t, s, x, y, z, w to T, S, x, y, z, w . Then this system can also be written in the new variables T, S, x, y, z, w as the Hamiltonian system

$$\begin{aligned}
dx &= \left(-x^2 + y + w - \frac{T}{2}\right)dT + \left(-\frac{(x - z)(xw - zw - \alpha_1)}{S} + \frac{w}{2}\right)dS, \\
dy &= (2xy + \alpha_3)dT + \left(\frac{2xyw - 2yzw - \alpha_1y + \alpha_3w}{S}\right)dS, \\
dz &= \left(-z^2 + w + y - \frac{S}{2} - \frac{T}{2}\right)dT + \left(-z^2 + w - \frac{S}{2} - \frac{T}{2} + \frac{y}{2} - \frac{(X - Z)(XY - YZ + \alpha_3)}{S}\right)dS, \\
dw &= (2zw + \alpha_1)dT + \left(2zw + \alpha_1 - \frac{2xyw - 2yzw - \alpha_1y + \alpha_3w}{S}\right)dS
\end{aligned} \quad (62)$$

with the polynomial Hamiltonians

$$H_1 = -x^2y + \frac{y^2}{2} - \frac{Ty}{2} - \alpha_3x - z^2w + \frac{w^2}{2} - \frac{Sw}{2} - \alpha_1z - \frac{Tw}{2} + yw, \quad (63)$$

$$\begin{aligned}
H_2 &= -\frac{(x - z)(xw - zw - \alpha_1)y}{S} + \frac{yw}{2} - \frac{Tw}{2} - \frac{\alpha_3xw}{S} + \frac{\alpha_3zw}{S} \\
&\quad - z^2w + \frac{w^2}{2} - \frac{Sw}{2} - \alpha_1z.
\end{aligned} \quad (64)$$

We remark that K. Kimura obtained the Hamiltonian $H_1|_{S \rightarrow 0}$ by certain reduction of the Drinfeld-Sokolov hierarchy (see [10]).

13 A generalization of the system (57) to three variables

In this section, we present a generalization of the system (57) to three variables, which is equivalent to the polynomial Hamiltonian system

$$\begin{aligned} dx &= \frac{\partial H_1}{\partial y} dt + \frac{\partial H_2}{\partial y} ds + \frac{\partial H_3}{\partial y} du, & dy &= -\frac{\partial H_1}{\partial x} dt - \frac{\partial H_2}{\partial x} ds - \frac{\partial H_3}{\partial x} du, \\ dz &= \frac{\partial H_1}{\partial w} dt + \frac{\partial H_2}{\partial w} ds + \frac{\partial H_3}{\partial w} du, & dw &= -\frac{\partial H_1}{\partial z} dt - \frac{\partial H_2}{\partial z} ds - \frac{\partial H_3}{\partial z} du, \\ dq &= \frac{\partial H_1}{\partial p} dt + \frac{\partial H_2}{\partial p} ds + \frac{\partial H_3}{\partial p} du, & dp &= -\frac{\partial H_1}{\partial q} dt - \frac{\partial H_2}{\partial q} ds - \frac{\partial H_3}{\partial q} du \end{aligned} \quad (65)$$

with the symmetric Hamiltonians $H_i \in \mathbb{C}(t, s, u)[x, y, z, w, q, p]$ ($i = 1, 2, 3$)

$$\begin{aligned} H_1 &= H_{II}(x, y, t; \alpha_3) + R(x, y, z, w, t, s; \alpha_3, \alpha_1) + R(x, y, q, p, t, u; \alpha_3, \alpha_4) \\ H_2 &= \pi(H_1), \quad H_3 = (\pi \circ \pi)(H_1) \quad (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1), \end{aligned} \quad (66)$$

where the transformation π is explicitly given by

$$\pi : (*) \rightarrow (z, w, q, p, x, y, s, u, t; \alpha_3, \alpha_2, \alpha_4, \alpha_1). \quad (67)$$

Here the symbol $(*)$ denotes $(*) := (x, y, z, w, q, p, t, s, u; \alpha_1, \alpha_2, \alpha_3, \alpha_4)$, and the symbol

$R(q_l, p_l, q_m, p_m, t_l, t_m; \alpha, \beta)$ is explicitly given by

$$\begin{aligned} &R(q_l, p_l, q_m, p_m, t_l, t_m; \alpha, \beta) \\ &= \frac{\beta}{t_l - t_m} q_l p_l - \frac{\beta}{t_l - t_m} p_l q_m - \frac{\alpha}{t_l - t_m} q_l p_m + \frac{\alpha}{t_l - t_m} q_m p_m - \frac{\{2(q_l - q_m)^2 - t_l + t_m\} p_l p_m}{2(t_l - t_m)}. \end{aligned}$$

Theorem 13.1. *Let us consider a polynomial Hamiltonian system with Hamiltonian*

$H_i \in \mathbb{C}(t, s, u)[x, y, z, w, q, p]$ ($i = 1, 2, 3$). *We assume that*

(A1) $\deg(H_i) = 5$ *with respect to* x, y, z, w, q, p .

(A2) *This system becomes again a polynomial Hamiltonian system in each coordinate system* $\{U_j, (x_j, y_j, z_j, w_j, q_j, p_j)\}$ ($j = 1, \dots, 4$):

$$U_j = \mathbb{C}^6 \ni (x_j, y_j, z_j, w_j, q_j, p_j) \quad (j = 1, \dots, 4),$$

via the following birational and symplectic transformations

$$\begin{aligned} 1) x_1 &= \frac{1}{x}, \quad y_1 = -(yx + \alpha_3)x, \quad z_1 = z, \quad w_1 = w, \quad q_1 = q, \quad p_1 = p, \\ 2) x_2 &= \frac{1}{x}, \quad y_2 = -\{(y + w + p - 2x^2 - t)x - 2\left((z - x)x - \frac{t - s}{4}\right)\left(\frac{w}{x}\right) \\ &\quad - 2\left((q - x)x - \frac{t - u}{4}\right)\left(\frac{p}{x}\right) + \alpha_2\}x, \\ z_2 &= \left((z - x)x - \frac{t - s}{2}\right)x, \quad w_2 = \frac{w}{x^2}, \quad q_2 = \left((q - x)x - \frac{t - u}{2}\right)x, \quad p_2 = \frac{p}{x^2}, \\ 3) x_3 &= x, \quad y_3 = y, \quad z_3 = \frac{1}{z}, \quad w_3 = -(zw + \alpha_1)z, \quad q_3 = q, \quad p_3 = p, \\ 4) x_4 &= x, \quad y_4 = y, \quad z_4 = z, \quad w_4 = w, \quad q_4 = \frac{1}{q}, \quad p_4 = -(qp + \alpha_4)q. \end{aligned}$$

Then such a system coincides with the system (65).

Theorem 13.2. *The system (65) is invariant under the following transformations: with the notation $(*) = (x, y, z, w, q, p, t, s, u; \alpha_0, \alpha_1, \dots, \alpha_4)$,*

$$\begin{aligned} s_1 : (*) &\rightarrow \left(x + \frac{\alpha_3}{y}, y, z, w, q, p, t, s, u; \alpha_1, \alpha_2 + \alpha_3, -\alpha_3, \alpha_4\right), \\ s_3 : (*) &\rightarrow \left(x, y, z + \frac{\alpha_1}{w}, w, q, p, t, s, u; -\alpha_1, \alpha_2 + \alpha_1, \alpha_3, \alpha_4\right), \\ s_4 : (*) &\rightarrow \left(x, y, z, w, q + \frac{\alpha_4}{p}, p, t, s, u; \alpha_1, \alpha_2 + \alpha_4, \alpha_3, -\alpha_4\right), \\ \pi_1 : (*) &\rightarrow (z, w, q, p, x, y, s, u, t; \alpha_3, \alpha_2, \alpha_4, \alpha_1). \end{aligned}$$

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